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# Analysis of the dynamics of avian influenza A(H7N9) epidemic model with re-infection

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**Abstract:** Since the emergence of the avian influenza A(H7N9) in the year 2013 in China, several researches have been carried out to investigate the spread. In this paper, a mathematical model describing the transmission dynamics of avian influenza A(H7N9) between human and poultry proposed by Li *et al.* [1] is modified by introducing re-infections into the susceptible human compartment. The method of next generation matrix is used to calculate the reproduction number. We also establish the local and global stability of the equilibria using Lyapunov functions. Finally, we use numerical simulations to validate our results.

**Keywords:** Avian influenza A(H7N9), reproduction number, Lyapunov functions, next generation matrix, re-infections.

**MSC:** 34D20, 34D20, 65P40, 37N25.

## 1. Introduction

Avian influenza refers to a group of viruses that are spread between birds and are grouped into Low Pathogenicity Avian Influenza (LPAI) and Highly Pathogenicity Avian Influenza (HPAI). One novel LPAI is the avian influenza A(H7N9) which emerged in humans in Eastern China in February 2013 [2–8]. The transmission of avian influenza A(H7N9) occurs between poultry and poultry or between poultry and human but no case has been recorded yet about human to human transmission. Iwami *et al.* in [9] considered a dynamic model of avian influenza that might be transmitted by infected poultry and infected humans with variant avian influenza. Che *et al.* [10] studied a model of highly pathogenic avian influenza with saturated contact rate. Liu *et al.* [11] established a dynamical model of avian influenza A(H7N9) that can spread between poultry and poultry, poultry and human, and human and human to evaluate the impact of these measures on avian influenza A(H7N9) epidemic. From the available epidemiological and virological evidence, thus the likelihood of human-to-human transmission of the avian influenza A(H7N9) is low. Li *et al.* [1] investigated the mathematical model describing the transmission dynamics of avian influenza A(H7N9) between human and poultry, in which there is no human-to-human transmission. They however, ignored the possibility of human re-infection. In [12], Khan *et al.* proposed two models considering saturated incidence rate and psychological effect in the model. Zhang *et al.* [13] constructed an avian influenza A (H7N9) epidemic model with vaccination and seasonality to study the spread of avian influenza A (H7N9). Other works on influenza virus include the works of [14–16]. In this paper, we extend the model proposed by [1]. We consider the possibility of human re-infection which transfers every recovered human back into the susceptible class.

## 2. The model

According to [1], human have the risk of getting infected with avian influenza A(H7N9) largely because of their exposure to poultry. In [1], Li *et al.* further developed a mathematical model to investigate the spread of avian influenza A(H7N9) to human. The assumptions in the model include:

1. transmission of avian influenza A(H7N9) occurs only in two ways; firstly, between poultry and poultry and secondly, between poultry and human. It cannot transmit from human to human.

2. the human population is classified each into Susceptible, Infected, and Recovered.
3. poultry in the farm and poultry in the market are classified into Susceptible and Infected.

Consider  $S$ ,  $I$ , and  $R$  denote Susceptible, Infected, and Recovered class respectively and the subscripts  $h$ ,  $fa$ , and  $ma$  denote human, farm and market respectively. The established model is given as:

$$\left. \begin{aligned} \frac{dS_{fa}}{dt} &= A_a - d_a S_{fa} - a S_{fa} - \beta_a S_{fa} I_{fa} \\ \frac{dI_{fa}}{dt} &= \beta_a S_{fa} I_{fa} - d_a I_{fa} - a I_{fa} - \alpha_a I_{fa} \\ \frac{dS_{ma}}{dt} &= a S_{fa} - \beta_m S_{ma} I_{ma} - d_m S_{ma} \\ \frac{dI_{ma}}{dt} &= a I_{fa} + \beta_m S_{ma} I_{ma} - d_m I_{ma} - \alpha_m I_{ma} \\ \frac{dS_h}{dt} &= A_h - \beta_h S_h I_{ma} - d_h S_h \\ \frac{dI_h}{dt} &= \beta_h S_h I_{ma} - d_h I_h - r I_h - \alpha_h I_h \\ \frac{dR_h}{dt} &= r I_h - d_h R_h. \end{aligned} \right\} \quad (1)$$

where  $A_h$  and  $A_a$  represent the birth rates of human and poultry respectively, and  $d_h$ ,  $d_a$  and  $d_m$  indicate the natural mortality rates of human, poultry of farms, and poultry of markets respectively.

Here  $\alpha_h$ ,  $\alpha_a$  and  $\alpha_m$  are the disease-related death rates of infected human, infected poultry of farms, and infected poultry of markets respectively,  $\beta_a$  is the transmission coefficient from infective poultry of farms to susceptible poultry of farms,  $\beta_m$  is the contact rate from infective poultry of markets to susceptible poultry of markets,  $\beta_h$  is the transmission rate from infected poultry of markets to susceptible human,  $r$  is the recovery rate of infected human and  $a$  is the proportion of poultry from farms to markets. All the parameters are non-negative.

It is worth noting that this model ignores the possibility of re-infection. We modify this model by assuming that a recovered individual who comes in contact with infective poultry can become infected again. This implies that any recovered individual moves out of the recovered compartment and thus, the equation for the recovered compartment becomes:

$$\frac{dR_h}{dt} = r I_h - R_h.$$

It is also important to note that some of the recovered individual may die naturally. This means that the population of the recovered individuals entering back into the Susceptible class is  $(1 - d_h) R_h$  and the equation for the Susceptible class becomes

$$\frac{dS_h}{dt} = A_h - \beta_h S_h I_{ma} - d_h S_h + (1 - d_h) R_h.$$

Combining these equations with the existing ones gives the proposed model as

$$\frac{dS_{fa}}{dt} = A_a - d_a S_{fa} - a S_{fa} - \beta_a S_{fa} I_{fa} \quad (2)$$

$$\frac{dI_{fa}}{dt} = \beta_a S_{fa} I_{fa} - d_a I_{fa} - a I_{fa} - \alpha_a I_{fa} \quad (3)$$

$$\frac{dS_{ma}}{dt} = a S_{fa} - \beta_m S_{ma} I_{ma} - d_m S_{ma} \quad (4)$$

$$\frac{dI_{ma}}{dt} = a I_{fa} + \beta_m S_{ma} I_{ma} - d_m I_{ma} - \alpha_m I_{ma} \quad (5)$$

$$\frac{dS_h}{dt} = A_h - \beta_h S_h I_{ma} - d_h S_h + (1 - d_h) R_h \quad (6)$$

$$\frac{dI_h}{dt} = \beta_h S_h I_{ma} - d_h I_h - r I_h - \alpha_h I_h \quad (7)$$

$$\frac{dR_h}{dt} = r I_h - R_h. \quad (8)$$

## 2.1. Non-negativity Conditions

In order to establish the validity of the model, we establish the non-negativity conditions for the farm poultry population, market poultry population and the human population. Let

$$N_{fa}(t) = S_{fa}(t) + I_{fa}(t) \quad (9)$$

$$N_{ma}(t) = S_{ma}(t) + I_{ma}(t) \quad (10)$$

$$N_h(t) = S_h(t) + I_h(t) + R_h(t) \quad (11)$$

where  $N_{fa}$ ,  $N_{ma}$  and  $N_h$  represent the total population of poultry in farm, total population of poultry in market and total human population respectively. By putting (9-11) into (2-8), Equations (2-8) reduce to the following system of three linear ordinary differential equations

$$\frac{dN_{fa}}{dt} = A_a - (d_a + a)N_{fa} - \alpha_a I_{fa} \leq A_a - (d_a + a)N_{fa} \quad (12)$$

$$\frac{dN_{ma}}{dt} = aN_{fa} - d_m N_{ma} - \alpha_m I_{ma} \leq aN_{fa} - d_m N_{ma}, \quad (13)$$

$$\frac{dN_h}{dt} = A_h - d_h N_h - \alpha_h I_h \leq A_h - d_h N_h \quad (14)$$

Then, from (12), it follows that

$$N_{fa}(t) \leq \frac{A_a}{a + d_a} + \left( N_{fa}(0) - \frac{A_a}{a + d_a} \right) e^{-(a+d_a)t} \quad (15)$$

and so

$$\lim_{t \rightarrow \infty} N_{fa}(t) \leq \frac{A_a}{a + d_a}.$$

In the same way, it can be obtained from (13) and (14) that

$$\lim_{t \rightarrow \infty} N_{ma}(t) \leq \frac{aA_a}{d_m(a + d_a)}, \text{ and } \lim_{t \rightarrow \infty} N_h(t) \leq \frac{A_h}{d_h}.$$

The feasible region of the system (2-8) is

$$\Omega = \left\{ (S_{fa}, I_{fa}, S_{ma}, I_{ma}, S_h, I_h, R_h) \in \mathbb{R}_+^7 : N_{fa} \leq \frac{A_a}{a + d_a}, N_{ma} \leq \frac{aA_a}{d_m(a + d_a)}, N_h \leq \frac{A_h}{d_h} \right\}.$$

## 3. The Existence of Equilibria

The reproduction number is calculated in this section and we further show the existence of the positive equilibrium. It is easy to see from (2-8), that the disease-free equilibrium is  $E^0 = (S_{fa}^0, 0, S_{ma}^0, 0, S_h^0, 0, 0)$ , where

$$S_{fa}^0 = \frac{A_a}{a + d_a}, S_{ma}^0 = \frac{aA_a}{d_m(a + d_a)}, \text{ and } S_h^0 = \frac{A_h}{d_h}. \quad (16)$$

Using the method proposed by [17], we can obtain the reproduction number by the next generation matrix

$$R_0 = \rho(FV^{-1})$$

where  $R_0$  is the reproduction number,  $\rho$  is the spectral radius of a matrix and

$$F = \left[ \frac{\partial f_i}{\partial x_i}(x_0) \right] \text{ and } V = \left[ \frac{\partial v_j}{\partial x_j}(x_0) \right] \text{ and } x_i \text{ are the relevant variables.}$$

For the problem at hand, we have

$$f = \begin{pmatrix} \beta_a S_{fa} I_{fa} \\ \beta_m S_{ma} I_{ma} \\ \beta_h S_h I_{ma} \end{pmatrix}, \quad v = \begin{pmatrix} d_a I_{fa} + a I_{fa} + \alpha_a I_{fa} \\ d_m I_{ma} + \alpha_m I_{ma} - a I_{fa} \\ d_h I_h + r I_h + \alpha_h I_h \end{pmatrix}$$

and consequently, we get reproduction number as

$$R_{01} = \frac{A_a \beta_a}{(a + d_a)(a + d_a + \alpha_a)}, R_{02} = \frac{a A_a \beta_m}{d_m(a + d_a)(d_m + \alpha_m)}.$$

Hence, the basic reproduction number of the system is

$$R_0 = \max\{R_{01}, R_{02}\}.$$

**Theorem 1.** For system (2-8),

1. If  $R_{01} > 1, R_{02} > 1$ , there is the unique endemic equilibrium

$$E^{**} = (S_{fa}^{**}, I_{fa}^{**}, S_{ma}^{**}, I_{ma}^{**}, S_h^{**}, I_h^{**}, R_h^{**}).$$

2. If  $R_{01} < 1, R_{02} > 1$ , then there is the unique boundary equilibrium

$$E^* = (S_{fa}^*, 0, S_{ma}^*, S_h^*, I_h^*, R_h^*).$$

**Proof.** There are only two possibilities for positive equilibrium. The first is when there are infective farm poultry (i.e.  $I_{fa} \neq 0$ ) while the second is when there are no infective farm poultry ( $I_{fa} = 0$ ).

1. Consider the first case where  $I_{fa} \neq 0$ , then it is obtained from equation (3) that

$$S_{fa}^{**} = \frac{a + \alpha_a + d_a}{\beta_a}. \tag{17}$$

Putting (17) in (2), we have

$$I_{fa}^{**} = \frac{(a + d_a)(R_{01} - 1)}{\beta_a}. \tag{18}$$

Since all parameters are positive and it is a necessary condition that  $I_{fa}^{**} > 0$ , then we require  $R_{01} > 1$  (from 18). Furthermore, suppose  $I_{ma} = 0$ , then Equation (5) implies that  $I_{fa} = 0$  which contradicts the initial assumption. Hence, we are left to take  $I_{ma} \neq 0$  and Equation (4) gives

$$S_{ma}^{**} = \frac{a S_{fa}^{**}}{\beta_m I_{ma}^{**} + d_m}. \tag{19}$$

Substituting (19) into (5) gives

$$a_1 I_{ma}^{**2} + b_1 I_{ma}^{**} + c_1 = 0, \tag{20}$$

where

$$\begin{aligned} a_1 &= -\beta_m (d_m + \alpha_m) < 0, \\ b_1 &= a\beta_m I_{fa}^{**} + a\beta_m S_{fa}^{**} - d_m^2 - d_m \alpha_m, \\ c_1 &= a d_m I_{fa}^{**} > 0, \end{aligned} \tag{21}$$

which means that  $\Delta = b_1^2 - 4a_1c_1 > 0$  if  $R_{02} > 1$  and hence, there is a unique positive root

$$I_{ma}^{**} = \frac{-a I_{fa}^{**}}{\beta_m S_{ma}^{**} - d_m - \alpha_m} = \frac{-b_1 - \sqrt{b_1^2 - 4a_1c_1}}{2a_1}. \tag{22}$$

Substituting (22) into Equations (6-8) gives

$$S_h^{**} = \frac{A_h + (1 - d_h)R_h^{**}}{\beta_h I_{ma}^{**} + d_h}, I_h^{**} = \frac{\beta_h S_h^{**} I_{ma}^{**}}{d_h + r + \alpha_h}, R_h^{**} = r I_h^{**}. \tag{23}$$

Therefore, we conclude that when  $R_{01} > 1$ , and  $R_{02} > 1$ , the endemic equilibrium

$$E^{**} = (S_{fa}^{**}, I_{fa}^{**}, S_{ma}^{**}, I_{ma}^{**}, S_h^{**}, I_h^{**}, R_h^{**})$$

is obtained.

- We now suppose  $I_{fa} = 0$  and this means Equation (3) becomes redundant. We obtain from Equation (2) that

$$S_{fa}^* = \frac{A_a}{d_a + a} \tag{24}$$

Substituting  $I_{fa} = 0$  into Equation (5) gives that either  $I_{ma} = 0$  or  $S_{ma} = \frac{d_m + \alpha_m}{\beta_m}$  but since  $I_{ma} \neq 0$  then we are left with

$$S_{ma}^* = \frac{d_m + \alpha_m}{\beta_m}. \tag{25}$$

Putting (24) and (25) into (4), we have

$$I_{ma}^* = \frac{aS_{fa}^* - d_m S_{ma}^*}{\beta_m S_{ma}^*} = \frac{aA_a}{(d_m + \alpha_m)(a + d_a)} - \frac{d_m}{\beta_m} = \frac{d_m(R_{02} - 1)}{\beta_m}, \tag{26}$$

from which it is required that  $R_{02} > 1$ . Substituting (26) into Equations (6-8), we finally have

$$S_h^* = \frac{A_h + (1 - d_h)R_h^*}{\beta_h I_{ma}^* + d_h}, I_h^* = \frac{\beta_h S_h^* I_{ma}^*}{d_h + r + \alpha_h}, R_h^* = r I_h^* \tag{27}$$

where  $R_{01} < 1$ . Hence, when  $R_{01} < 1$  and  $R_{02} > 1$ , the boundary equilibrium

$$E^* = (S_{fa}^*, 0, S_{ma}^*, S_h^*, I_h^*, R_h^*)$$

is obtained.

□

### 4. Stability of equilibria

It has been established in the previous section that there exists disease-free and positive equilibria. We further investigate the stability of this equilibria.

#### 4.1. Stability of the disease-free equilibrium

**Theorem 2.** Let  $E^0 = (S_{fa}^0, 0, S_{ma}^0, 0, S_h^0, 0, 0)$  be the disease-free equilibrium of system (2-8).

- If  $R_0 < 1$ , then  $E^0$  is locally asymptotically stable;
- If  $R_0 > 1$ , then  $E^0$  is unstable.

**Proof.** The Jacobian matrix at the disease-free equilibrium  $E^0$  is

$$J_{E^0} = \begin{pmatrix} -d_a - a & -\beta_a S_{fa}^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_a S_{fa}^0 - d_a - a - \alpha_a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & -d_m & -\beta_a S_{ma}^0 & 0 & 0 & 0 \\ 0 & a & 0 & \beta_m S_{ma}^0 - d_m - \alpha_m & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_h S_h^0 & -d_h & 0 & 1 - d_h \\ 0 & 0 & 0 & \beta_h S_h^0 & 0 & -d_h - \alpha_h - r & 0 \\ 0 & 0 & 0 & 0 & 0 & r & -1 \end{pmatrix} \tag{28}$$

The characteristics equation of the Jacobian matrix is

$$(\lambda + 1) (\lambda + d_h) (\lambda + d_m) (\lambda + d_a + a) (\lambda + d_h + r + \alpha_h) \times (\lambda + d_a + a + \alpha_a - \beta_a S_{fa}^0) (\lambda + \alpha_m + d_m - \beta_m S_{ma}^0) = 0,$$

and the eigenvalues are

$$\begin{aligned}\lambda_1 &= -1, \lambda_2 = -d_h, \lambda_3 = -d_m, \lambda_4 = -(d_a + a), \lambda_5 = -(d_h + \alpha_h + r), \\ \lambda_6 &= -(d_a + a + \alpha_a) + \frac{\beta_a A_a}{a + d_a} = (R_{01} - 1)(d_a + a + \alpha_a) < 0 \text{ if } R_{01} < 1, \\ \lambda_7 &= -(d_m + \alpha_m) + \frac{\beta_m a A_a}{d_m(a + d_a)} = (R_{02} - 1)(d_m + \alpha_m) < 0 \text{ if } R_{02} < 1.\end{aligned}$$

Hence, all eigenvalues have negative real parts if  $R_{01} < 1$  and  $R_{02} < 1$  and consequently,  $R_0 = \max\{R_{01}, R_{02}\} < 1$  (see [18]). Thus, the disease-free equilibrium  $E_0$  is locally asymptotically stable if  $R_0 < 1$ .  $\square$

**Theorem 3.** For system (5), if  $R_{01} < 1$ , the disease-free equilibrium  $E^0$  is globally asymptotically stable.

**Proof.** The proof shall be constructed by taking the three different subsystems one after the other

1. **Poultry subsystem in farms:** Define a Lyapunov function for the poultry subsystem in farms

$$V_{11} = S_{fa} - S_{fa}^0 - S_{fa}^0 \ln \left( \frac{S_{fa}}{S_{fa}^0} \right) + I_{fa},$$

and then the derivative of  $V_{11}$  is

$$\frac{dV_{11}}{dt} = \frac{S_{fa} - S_{fa}^0}{S_{fa}} \frac{dS_{fa}}{dt} + \frac{dI_{fa}}{dt} \quad (29)$$

and substituting (16), (2) and 3 into (29), we have

$$\frac{dV_{11}}{dt} = -\frac{a + d_a}{S_{fa}} (S_{fa} - S_{fa}^0)^2 + \beta_a S_{fa}^0 I_{fa} - (d_a + a + \alpha_a) I_{fa}, \quad (30)$$

$$= -\frac{a + d_a}{S_{fa}} (S_{fa} - S_{fa}^0)^2 + (d_a + a + \alpha_a) (R_{01} - 1) I_{fa}. \quad (31)$$

Clearly,  $\frac{dV_{11}}{dt} \leq 0$  if  $R_{01} < 1$ . We now have that

$$\Omega_1 = \left\{ (S_{fa}, I_{fa}) \in \mathbb{R}_+^2 : \frac{dV_{11}}{dt} = 0 \right\} = \left\{ (S_{fa}, I_{fa}) \in \mathbb{R}_+^2 : S_{fa} = S_{fa}^0, I_{fa} = 0 \right\} = \left\{ E_{fa}^0 \right\},$$

which according to Lassalet's invariance principle,  $E_{fa}^0$  is globally asymptotically stable [19,20].

2. **Poultry subsystem of markets:** Poultry subsystem of markets with the avian components of farms already at the disease-free steady state is

$$\frac{dS_{ma}}{dt} = aS_{fa}^0 - \beta_m S_{ma} I_{ma} - d_m S_{ma}, \quad (32)$$

$$\frac{dI_{ma}}{dt} = \beta_m S_{ma} I_{ma} - d_m I_{ma} - \alpha_m I_{ma}. \quad (33)$$

We define a Lyapunov function as

$$V_{12} = S_{ma} - S_{ma}^0 - S_{ma}^0 \ln \left( \frac{S_{ma}}{S_{ma}^0} \right) + I_{ma}.$$

It follows that

$$\frac{dV_{12}}{dt} = \frac{S_{ma} - S_{ma}^0}{S_{ma}} \frac{dS_{ma}}{dt} + \frac{dI_{ma}}{dt}$$

Replacing  $\frac{dS_{ma}}{dt}$  and  $\frac{dI_{ma}}{dt}$  with equations (32) and (33) respectively, then substituting  $S_{fa}^0 = \frac{A_a}{a+d_a}$  and  $d_m S_{ma}^0 = a \frac{A_a}{a+d_a}$  from 16 and rearranging, we have

$$\frac{dV_{12}}{dt} = -\frac{d_m}{S_{ma}} (S_{ma} - S_{ma}^0)^2 + (d_m + \alpha_m) (R_{02} - 1) I_{ma},$$

from which  $\frac{dV_{12}}{dt} \leq 0$  if  $R_{02} < 1$ . Thus,

$$\Omega_2 = \left\{ (S_{ma}, I_{ma}) \in \mathbb{R}_+^2 : \frac{dV_{12}}{dt} = 0 \right\} = \left\{ (S_{ma}, I_{ma}) \in \mathbb{R}_+^2 : S_{ma} = S_{ma}^0, I_{ma} = 0 \right\} = \left\{ E_{ma}^0 \right\}.$$

According to Lasalle’s invariance principle ([19,20]),  $E_{ma}^0$  is globally asymptotically stable.

3. **Human subsystem:** Finally, considering the human subsystem with the avian components already at the disease-free steady state

$$\left. \begin{aligned} \frac{dS_h}{dt} &= A_h - d_h S_h + R_h (1 - d_h), \\ \frac{dI_h}{dt} &= -d_h I_h - r I_h - \alpha_h I_h, \\ \frac{dR_h}{dt} &= r I_h - R_h, \end{aligned} \right\} \tag{34}$$

We define a Lyapunov function

$$V_{13} = S_h - S_h^0 - S_h^0 \ln \frac{S_h}{S_h^0} + I_h + R_h,$$

and then the derivative of  $V_{13}$  along the solutions of system (34) is

$$\frac{dV_{13}}{dt} = \frac{S_h - S_h^0}{S_h} \frac{dS_h}{dt} + \frac{dI_h}{dt} + \frac{dR_h}{dt} \tag{35}$$

Substituting  $A_h = d_h S_h^0$  and Equations (34) in Equation (35), we have

$$\frac{dV_{13}}{dt} = -\frac{d_h}{S_h} (S_h - S_h^0)^2 + \frac{S_h - S_h^0}{S_h} (1 - d_h) R_h - (d_h + \alpha_h) I_h - R_h \leq 0,$$

and, thus,  $\Omega_3 = \left\{ (S_h, I_h, R_h) \in \mathbb{R}_+^3 : \frac{dV_{13}}{dt} = 0 \right\} = \left\{ (S_h, I_h, R_h) \in \mathbb{R}_+^3 : S_h = S_h^0, I_h = 0, R_h = 0 \right\} = \left\{ E_h^0 \right\}$ . According to Lasalle’s invariance principle,  $E_h^0$  is globally asymptotically stable.

□

### 4.2. Stability of the boundary equilibrium and the endemic equilibrium

The Jacobian matrix of system (2-8) is given as,

$$J = \begin{pmatrix} -d_a - a - \beta_a I_{fa} & -\beta_a S_{fa} & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_a I_{fa} & \beta_a S_{fa} - d_a - a - \alpha_a & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & -\beta_m I_{ma} - d_m & -\beta_m S_{ma} & 0 & 0 & 0 & 0 \\ 0 & a & \beta_m I_{ma} & \beta_m S_{ma} - d_m - \alpha_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_h S_h & -\beta_h I_{ma} - d_h & 0 & 1 - d_h & 0 \\ 0 & 0 & 0 & \beta_h S_h & \beta_h I_{ma} & -d_h - r - \alpha_h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r & -1 & -1 \end{pmatrix}.$$

The characteristics equation of the Jacobian matrix is:

$$\begin{aligned} & \left[ (\lambda + d_a + a + \beta_a I_{fa}) (\lambda + d_a + a + \alpha_a - \beta_a S_{fa}) + \beta_a^2 S_{fa} I_{fa} \right] \\ & \times \left[ (\lambda + d_m + \beta_m I_{ma}) (\lambda + \alpha_m + d_m - \beta_m S_{ma}) + \beta_m^2 S_{ma} I_{ma} \right] \\ & \times [(\lambda + d_h + \beta_h I_{ma}) (\lambda + d_h + r + \alpha_h) (\lambda + 1)] = 0 \end{aligned} \tag{36}$$

**Theorem 4.** For system (5), if  $R_{01} < 1$ ,  $R_{02} > 1$ , the boundary equilibrium  $E^*$  is locally asymptotically stable; if  $R_{01} > 1$ ,  $R_{02} > 1$ , the endemic equilibrium  $E^{**}$  is locally asymptotically stable.

**Proof.** The proof is as follows;

1. For the boundary equilibrium  $E^* = (S_{fa}^*, 0, S_{ma}^*, I_{ma}^*, S_h^*, I_h^*, R_h^*)$ , five of the eigenvalues are

$$\begin{aligned}\lambda_1 &= -1, \lambda_2 = -d_h - r - \alpha_h, \lambda_3 = -d_h - \beta_h I_{ma}^*, \\ \lambda_4 &= -a - d_a, \lambda_5 = -\alpha_a - a - d_a + \frac{\beta_a A_a}{a + d_a} = (\alpha_a + a + d_a)(R_{01} - 1)\end{aligned}$$

and the remaining two eigenvalues  $\lambda_6, \lambda_7$  depend on

$$\lambda^2 + \lambda d_m (R_{02} - 1) + d_m (\alpha_m + d_m) (R_{02} - 1) = 0.$$

Hence, if  $R_{01} < 1$  and  $R_{02} > 1$ , all eigenvalues have negative real parts.

2. For the endemic equilibrium  $E^{**} = (S_{fa}^{**}, I_{fa}^{**}, S_{ma}^{**}, I_{ma}^{**}, S_h^{**}, I_h^{**}, R_h^{**})$ , three of the eigenvalues are

$$\lambda_1 = -1, \lambda_2 = -d_h - r - \alpha_h, \lambda_3 = -d_h - \beta_h I_{ma}^{**}$$

while  $\lambda_4, \lambda_5$  are obtained from

$$\lambda^2 + \lambda R_{01} (d_a + a) + (d_a + a) (\alpha + a + d_a) (R_{01} - 1) = 0$$

and the remaining two eigenvalues  $\lambda_6, \lambda_7$  are obtained from

$$\lambda^2 + \lambda (2d_m + \alpha_m - \beta_m S_{ma}^{**} + \beta_m I_{ma}^{**}) + \beta_m d_m I_{ma}^{**} + \beta_m \alpha_m I_{ma}^{**} + d_m (d_m + \alpha_m - \beta_m S_{ma}^{**}) = 0$$

It can be obtained that  $d_m + \alpha_m - \beta_m S_{ma}^{**} > 0$  by  $I_{ma}^{**} > 0$ . Hence, if  $R_{01} > 1, R_{02} > 1$ , all eigenvalues have negative real parts.

□

**Theorem 5.** For system (5),

1. if  $R_{01} < 1, R_{02} > 1$ , the boundary equilibrium  $E^*$  is globally asymptotically stable,
2. if  $R_{01} > 1, R_{02} > 1$ , the endemic equilibrium  $E^{**}$  is globally asymptotically stable.

**Proof.** We consider the global stability of the boundary equilibrium and the endemic equilibrium.

1. The boundary equilibrium  $E^*$ .

(a) Consider the poultry subsystem in farms and define a Lyapunov function

$$V_{21} = S_{fa} - S_{fa}^* - S_{fa}^* \ln \left( \frac{S_{fa}}{S_{fa}^*} \right) + I_{fa}$$

and then the derivative of  $V_{21}$  is

$$\frac{dV_{21}}{dt} = \frac{S_{fa} - S_{fa}^*}{S_{fa}} \frac{dS_{fa}}{dt} + \frac{dI_{fa}}{dt} \quad (37)$$

Substituting  $A = S_{fa}^* (d_a + a)$ , (2) and (3) into (37), we have

$$\frac{dV_{21}}{dt} = - \frac{(S_{fa} - S_{fa}^*)^2}{S_{fa}} (d_a + a) + \beta_a S_{fa}^* I_{fa} - (d_a + a + \alpha_a) I_{fa} \quad (38)$$

Substituting  $S_{fa}^*$  from (24) into the second term of the right hand side of (38)

$$\frac{dV_{21}}{dt} = - \frac{(d_a + a)}{S_{fa}} (S_{fa} - S_{fa}^*)^2 + (d_a + a + \alpha_a) (R_{01} - 1) I_{fa}$$

and if  $R_{01} < 1$ ,  $\frac{dV_{21}}{dt} \leq 0$ , then

$$\Omega_4 = \left\{ (S_{fa}, I_{fa}) \in \mathbb{R}_+^2 : \frac{dV_{21}}{dt} = 0 \right\} = \left\{ (S_{fa}, I_{fa}) \in \mathbb{R}_+^2 : S_{fa} = S_{fa}^*, I_{fa} = 0 \right\} = \{E_{fa}^*\}.$$

By Lassalle’s invariance principle,  $E_{fa}^*$  is globally asymptotically stable.

- (b) Next, considering the poultry subsystem of markets with the avian components of farms already at the disease-free steady state.

$$\frac{dS_{ma}}{dt} = aS_{fa}^* - \beta_m S_{ma} I_{ma} - d_m S_{ma} \tag{39}$$

$$\frac{dI_{ma}}{dt} = \beta_m S_{ma} I_{ma} - d_m I_{ma} - \alpha_m I_{ma}. \tag{40}$$

We define a Lyapunov function

$$V_{22} = S_{ma} - S_{ma}^* - S_{ma}^* \ln \left( \frac{S_{ma}}{S_{ma}^*} \right) + I_{ma} - I_{ma}^* - I_{ma}^* \left( \frac{I_{ma}}{I_{ma}^*} \right),$$

and then the derivative of  $V_{22}$  is

$$\frac{dV_{22}}{dt} = \frac{S_{ma} - S_{ma}^*}{S_{ma}} \frac{dS_{ma}}{dt} + \frac{I_{ma} - I_{ma}^*}{I_{ma}} \frac{dI_{ma}}{dt}, \tag{41}$$

and we can deduce that from (24) and (25) that

$$aS_{fa}^* = \beta_m S_{ma}^* I_{ma}^* + d_m S_{ma}^* \text{ and } d_m + \alpha_m = \beta_m S_{ma}^*. \tag{42}$$

Substituting (39) and (40) into (41), then using (42) and rearranging, we have

$$\frac{dV_{22}}{dt} = \left( 2 - \frac{S_{ma}^*}{S_{ma}} - \frac{S_{ma}}{S_{ma}^*} \right) (d_m S_{ma}^* + d_m S_{ma}^* (R_{02} - 1))$$

Since  $2 - \frac{S_{ma}^*}{S_{ma}} - \frac{S_{ma}}{S_{ma}^*} \leq 0$ , if  $R_{01} > 1$ ,  $\frac{dV_{22}}{dt} \leq 0$ , thus,

$$\begin{aligned} \Omega_5 &= \left\{ (S_{ma}, I_{ma}) \in \mathbb{R}_+^2 : dV_{22}/dt = 0 \right\} \\ &= \left\{ (S_{ma}, I_{ma}) \in \mathbb{R}_+^2 : S_{ma} = S_{ma}^*, I_{ma} = I_{ma}^* \right\} = \{E_{ma}^*\}. \end{aligned}$$

According to Lassalle’s invariance principle,  $E_{ma}^*$  is globally asymptotically stable.

- (c) Finally, considering the human subsystem with the avian components of markets already at the endemic steady state

$$\left. \begin{aligned} \frac{dS_h}{dt} &= A_h - \beta_h S_h I_{ma}^* - d_h S_h + (1 - d_h) R_h, \\ \frac{dI_h}{dt} &= \beta_h S_h I_{ma}^* - d_h I_h - r I_h - \alpha_h I_h, \\ \frac{dR_h}{dt} &= r I_h - R_h. \end{aligned} \right\} \tag{43}$$

We define a Lyapunov function

$$V_{23} = S_h - S_h^* - S_h^* \ln \frac{S_h}{S_h^*} + I_h - I_h^* - I_h^* \ln \frac{I_h}{I_h^*} + R_h - R_h^* - R_h^* \ln \frac{R_h}{R_h^*}$$

and then the derivative of  $V_{23}$  is

$$\frac{dV_{23}}{dt} = \frac{S_h - S_h^*}{S_h} \frac{dS_h}{dt} + \frac{I_h - I_h^*}{I_h} \frac{dI_h}{dt} + \frac{R_h - R_h^*}{R_h} \frac{dR_h}{dt}, \tag{44}$$

and from (27), we can deduce

$$\left. \begin{aligned} A_h &= \beta_h S_h^* I_{ma}^* + d_h S_h^* - (1 - d_h) R_h, \\ d_h + r + \alpha_h &= \beta_h S_h^* \frac{I_{ma}^*}{I_h^*}, \\ r &= \frac{R_h^*}{I_h^*}. \end{aligned} \right\} \tag{45}$$

Substituting (43) into (44), then using (45) and rearranging, we have that

$$\begin{aligned} \frac{dV_{23}}{dt} &= d_h S_h^* \left( 2 - \frac{S_h^*}{S_h} - \frac{S_h}{S_h^*} + \frac{R_h}{S_h} - \frac{R_h}{S_h^*} + \frac{R_h^*}{S_h^*} - \frac{R_h^*}{S_h} \right) \\ &+ \beta_h S_h^* I_{ma}^* \left( 3 - \frac{S_h^*}{S_h} - \frac{I_h}{I_h^*} - \frac{S_h I_h^*}{S_h^* I_h} \right) + R_h^* \left( \frac{S_h^*}{S_h} - \frac{S_h^* R_h}{S_h R_h^*} - \frac{R_h^* I_h}{R_h I_h^*} + \frac{I_h}{I_h^*} \right). \end{aligned} \tag{46}$$

Substituting  $I_{ma}^*$  from (26) into (46), we have

$$\begin{aligned} \frac{dV_{23}}{dt} &= d_h S_h^* \left( 2 - \frac{S_h^*}{S_h} - \frac{S_h}{S_h^*} + \frac{R_h}{S_h} - \frac{R_h}{S_h^*} + \frac{R_h^*}{S_h^*} - \frac{R_h^*}{S_h} \right) \\ &+ \frac{d_m(R_{02} - 1)}{\beta_m} \beta_h S_h^* \left( 3 - \frac{S_h^*}{S_h} - \frac{I_h}{I_h^*} - \frac{S_h I_h^*}{S_h^* I_h} \right) \\ &+ R_h^* \left( \frac{S_h^*}{S_h} - \frac{S_h^* R_h}{S_h R_h^*} - \frac{R_h^* I_h}{R_h I_h^*} + \frac{I_h}{I_h^*} \right). \end{aligned}$$

Since

$$\begin{aligned} 2 - \frac{S_h^*}{S_h} - \frac{S_h}{S_h^*} + \frac{R_h}{S_h} - \frac{R_h}{S_h^*} + \frac{R_h^*}{S_h^*} - \frac{R_h^*}{S_h} &\leq 0, \\ 3 - \frac{S_h^*}{S_h} - \frac{I_h}{I_h^*} - \frac{S_h I_h^*}{S_h^* I_h} &\leq 0, \\ \frac{S_h^*}{S_h} - \frac{S_h^* R_h}{S_h R_h^*} - \frac{R_h^* I_h}{R_h I_h^*} + \frac{I_h}{I_h^*} &\leq 0 \end{aligned}$$

if  $R_{01} > 1$ ,  $\frac{dV_{23}}{dt} \leq 0$ , thus,

$$\begin{aligned} \Omega_6 &= \left\{ (S_h, I_h, R_h) \in \mathbb{R}_+^3 : \frac{dV_{23}}{dt} = 0 \right\} \\ &= \left\{ (S_h, I_h, R_h) \in \mathbb{R}_+^3 : S_h = S_h^*, I_h = I_h^*, R_h = R_h^* \right\} = \{E_h^*\}. \end{aligned}$$

According to Lassalle’s invariance principle,  $E_h^*$  is globally asymptotically stable. In conclusion, if  $R_{01} < 1$ ,  $R_{02} > 1$ , the boundary equilibrium  $E^*$  is globally asymptotically stable.

2. The endemic equilibrium  $E^{**}$ .

(a) Consider the poultry subsystem in farms and define a Lyapunov function

$$V_{31} = S_{fa} - S_{fa}^{**} - S_{fa}^{**} \ln \frac{S_{fa}}{S_{fa}^{**}} + I_{fa} - I_{fa}^{**} - I_{fa}^{**} \ln \frac{I_{fa}}{I_{fa}^{**}},$$

and then the derivative of  $V_{31}$  along solutions of system (5) is

$$\frac{dV_{31}}{dt} = \frac{S_{fa} - S_{fa}^{**}}{S_{fa}} \frac{dS_{fa}}{dt} + \frac{I_{fa} - I_{fa}^{**}}{I_{fa}} \frac{dI_{fa}}{dt}. \tag{47}$$

We can deduce that  $a + \alpha_a + d_a = \beta_a S_{fa}^{**}$  and  $A_a = \beta_a S_{fa}^{**} I_{fa}^{**} + (a + d_a) S_{fa}^{**}$  from (17) and (18) and on substituting into (47), we obtain

$$\frac{dV_{31}}{dt} = -\frac{(a + d_a)}{S_{fa}} (S_{fa} - S_{fa}^{**})^2 + \beta_a S_{fa}^{**} I_{fa}^{**} \left( 2 - \frac{S_{fa}^*}{S_{fa}} - \frac{S_{fa}}{S_{fa}^{**}} \right). \tag{48}$$

Substituting  $I_{fa}^{**}$  from (18) into (48) we have,

$$\frac{dV_{31}}{dt} = -\frac{(a + d_a)}{S_{fa}} (S_{fa} - S_{fa}^{**})^2 + S_{fa}^{**} (a + d_a) (R_{01} - 1) \left( 2 - \frac{S_{fa}^*}{S_{fa}} - \frac{S_{fa}}{S_{fa}^{**}} \right).$$

If  $R_{01} > 1$ , then  $2 - \frac{S_{fa}^{**}}{S_{fa}} - \frac{S_{fa}}{S_{fa}^{**}} \leq 0$  and  $\frac{dV_{31}}{dt} \leq 0$  and thus,

$$\begin{aligned} \Omega_7 &= \left\{ (S_{fa}, I_{fa}) \in \mathbb{R}_+^2 : \frac{dV_{31}}{dt} = 0 \right\} \\ &= \left\{ (S_{fa}, I_{fa}) \in \mathbb{R}_+^2 : S_{fa} = S_{fa}^{**}, I_{fa} = I_{fa}^{**} \right\} = \left\{ E_{fa}^{**} \right\}. \end{aligned}$$

- According to Lassalle’s invariance principle,  $E_h^{**}$  is globally asymptotically stable.  
 (b) Consider the poultry subsystem of markets with the avian components of farms already at the endemic steady state;

$$\frac{dS_{ma}}{dt} = aS_{fa}^{**} - \beta_m S_{ma} I_{ma} - d_m S_{ma} \tag{49}$$

$$\frac{dI_{ma}}{dt} = aI_{fa}^{**} + \beta_m S_{ma} I_{ma} - d_m I_{ma} - \alpha_m I_{ma} \tag{50}$$

we define a Lyapunov function

$$V_{32} = S_{ma} - S_{ma}^{**} - S_{ma}^{**} \ln \frac{S_{ma}}{S_{ma}^{**}} + I_{ma} - I_{ma}^{**} - I_{ma}^{**} \ln \frac{I_{ma}}{I_{ma}^{**}},$$

and then the derivative of  $V_{32}$  is

$$\frac{dV_{32}}{dt} = \frac{S_{ma} - S_{ma}^{**}}{S_{ma}} \frac{dS_{ma}}{dt} + \frac{I_{ma} - I_{ma}^{**}}{I_{ma}} \frac{dI_{ma}}{dt}.$$

Substitute Equations (49) and (50). Putting the relation  $aS_{fa}^{**} = d_m S_{ma}^{**} + \beta_m S_{ma}^{**} I_{ma}^{**}$  and  $d_m + \alpha_m = \beta_m S_{ma}^{**} + \frac{aI_{fa}^{**}}{I_{ma}^{**}}$  (which can be easily obtained from (17) and (22)) and substituting  $I_{fa}^{**}$  from (18) and then rearranging, we have

$$\begin{aligned} \frac{dV_{32}}{dt} &= \frac{a(a + d_a)(R_{01} - 1)}{\beta_a} \left( 2 - \frac{I_{ma}^{**}}{I_{ma}} - \frac{I_{ma}}{I_{ma}^{**}} \right) \\ &\quad + d_m S_{ma}^{**} \left( 2 - \frac{S_{ma}^{**}}{S_{ma}} - \frac{S_{ma}}{S_{ma}^{**}} \right) + \beta_m S_{ma}^{**} I_{ma}^{**} \left( 2 - \frac{S_{ma}^{**}}{S_{ma}} - \frac{S_{ma}}{S_{ma}^{**}} \right) \end{aligned}$$

Clearly, if  $R_{01} > 1$ ,  $R_{02} > 1$ , then

$$2 - \frac{S_{ma}^{**}}{S_{ma}} - \frac{S_{ma}}{S_{ma}^{**}} \leq 0, \quad 2 - \frac{I_{ma}^{**}}{I_{ma}} - \frac{I_{ma}}{I_{ma}^{**}} \leq 0$$

and  $\frac{dV_{32}}{dt} \leq 0$ . Thus,

$$\begin{aligned} \Omega_8 &= \left\{ (S_{ma}, I_{ma}) \in \mathbb{R}_+^2 : \frac{dV_{32}}{dt} = 0 \right\} \\ &= \left\{ (S_{ma}, I_{ma}) \in \mathbb{R}_+^2 : S_{ma} = S_{ma}^{**}, I_{ma} = I_{ma}^{**} \right\} = \left\{ E_{ma}^{**} \right\}. \end{aligned}$$

- According to Lassalle’s invariance principle,  $E_{ma}^{**}$  is globally asymptotically stable.  
 (c) Finally, we consider the human subsystem with the avian components already at the endemic steady state

$$\left. \begin{aligned} \frac{dS_h}{dt} &= A_h - \beta_h S_h I_{ma}^{**} - d_h S_h, \\ \frac{dI_h}{dt} &= \beta_h S_h I_{ma}^{**} - d_h I_h - r I_h - \alpha_h I_h, \\ \frac{dR_h}{dt} &= r I_h - R_h \end{aligned} \right\} \tag{51}$$

We define a Lyapunov function

$$V_{33} = S_h - S_h^{**} - S_h^{**} \ln \frac{S_h}{S_h^{**}} + I_h - I_h^{**} - I_h^{**} \ln \frac{I_h}{I_h^{**}} + R_h - R_h^{**} - R_h^{**} \ln \frac{R_h}{R_h^{**}}$$

and then the derivative of  $V_{33}$  is given as

$$\frac{dV_{33}}{dt} = \frac{S_h - S_h^{**}}{S_h} \frac{dS_h}{dt} + \frac{I_h - I_h^{**}}{I_h} \frac{dI_h}{dt} + \frac{R_h - R_h^{**}}{R_h} \frac{dR_h}{dt}$$

Using the relations  $A_h = d_h S_h^{**} + \beta_h S_h^{**} I_h^{**} - (1 - d_h) R_h^{**}$ ,  $d_h + r + \alpha_h = \beta_h S_h^{**} \frac{I_h^{**}}{I_h^{**}}$  and  $r = R_h^{**} / I_h^{**}$  (which can be easily obtained from (23)), we have

$$\begin{aligned} \frac{dV_{33}}{dt} = & d_h S_h^{**} \left( 2 - \frac{S_h^{**}}{S_h} - \frac{S_h}{S_h^{**}} + \frac{R_h}{S_h} - \frac{R_h}{S_h^{**}} + \frac{R_h^{**}}{S_h^{**}} - \frac{R_h^{**}}{S_h} \right) \\ & + \beta_h S_h^{**} I_h^{**} \left( 3 - \frac{S_h^{**}}{S_h} - \frac{I_h}{I_h^{**}} - \frac{S_h I_h^{**}}{S_h^{**} I_h} \right) \\ & + R_h^{**} \left( \frac{S_h^{**}}{S_h} - \frac{S_h^{**} R_h}{S_h R_h^{**}} - \frac{R_h^{**} I_h}{R_h I_h^{**}} + \frac{I_h}{I_h^{**}} \right) \end{aligned}$$

Since if  $R_{01} > 1$ , then

$$\begin{aligned} 2 - \frac{S_h^{**}}{S_h} - \frac{S_h}{S_h^{**}} + \frac{R_h}{S_h} - \frac{R_h}{S_h^{**}} + \frac{R_h^{**}}{S_h^{**}} - \frac{R_h^{**}}{S_h} & \leq 0 \\ 3 - \frac{S_h^{**}}{S_h} - \frac{I_h}{I_h^{**}} - \frac{S_h I_h^{**}}{S_h^{**} I_h} & \leq 0 \\ \frac{S_h^{**}}{S_h} - \frac{S_h^{**} R_h}{S_h R_h^{**}} - \frac{R_h^{**} I_h}{R_h I_h^{**}} + \frac{I_h}{I_h^{**}} & \leq 0 \end{aligned}$$

and  $\frac{dV_{33}}{dt} \leq 0$ . Thus,

$$\begin{aligned} \Omega_9 &= \left\{ (S_h, I_h, R_h) \in R_+^3 : dV_{33}/dt = 0 \right\} \\ &= \left\{ (S_h, I_h, R_h) \in R_+^3 : S_h = S_h^{**}, I_h = I_h^{**}, R_h = R_h^{**} \right\} = \{E_h^{**}\}. \end{aligned}$$

According to Lassalle’s invariance principle,  $E_h^{**}$  is globally asymptotically stable.

□

**Remark 1.** Stability of the equilibrium depends on the Lyapunov functions. The quadratic form of Lyapunov functions is usually used in most references [21,22]. Let us take the function in this paper

$$f(x) = 1 - x + \ln x, \quad \forall x > 0,$$

we have

$$f(1) = 0, \quad f'(x) = \frac{1}{x} - 1,$$

thus,  $f(x) = 1 - x + \ln x \leq 0 \quad \forall x > 0$ , and the equality holds only when  $x = 1$ . Let  $x = \frac{S_{fa}}{S_{fa}^0} > 0$ , then  $S_{fa} - S_{fa}^0 - S_{fa}^0 \ln \left( \frac{S_{fa}}{S_{fa}^0} \right) \geq 0$ . Hence,  $V_{11}$  is a Lyapunov function. In a similar way,  $V_{12}, V_{13}, V_{21}, V_{22}, V_{23}, V_{31}, V_{32}, V_{33}$  are also Lyapunov functions.

### 5. Numerical Simulation

The existence of the disease free equilibrium, the boundary equilibrium, and the endemic equilibrium of system (2-8) have been established in the previous sections. In what follows, we simulate the stability of these equilibria by plotting the graphs of the system (2-8) for different values of the parameters corresponding to each of these equilibria. Figure 1 verifies that the disease dies out of the population and Figures 2 and 3 indicate that the disease is sustained in the population.

Tables (1-4) show the effect of disease-related death rates of infected poultry of markets  $\alpha_m$ , the disease-related death rates of infected poultry of farms  $\alpha_a$ , the transmission coefficient from infective poultry of farms to susceptible poultry of farms  $\beta_a$  and the contact rate from infective poultry of markets to susceptible poultry of markets  $\beta_m$  on the reproduction number. As can be seen in Tables 1 and 2, increase in  $\alpha_m$  and  $\alpha_a$  lead to decrease in  $R_{02}$  and  $R_{01}$  respectively. Increase in  $\beta_a$  and  $\beta_m$  lead to an increase in  $R_{01}$  and  $R_{02}$  respectively (see Tables 3 and 4).

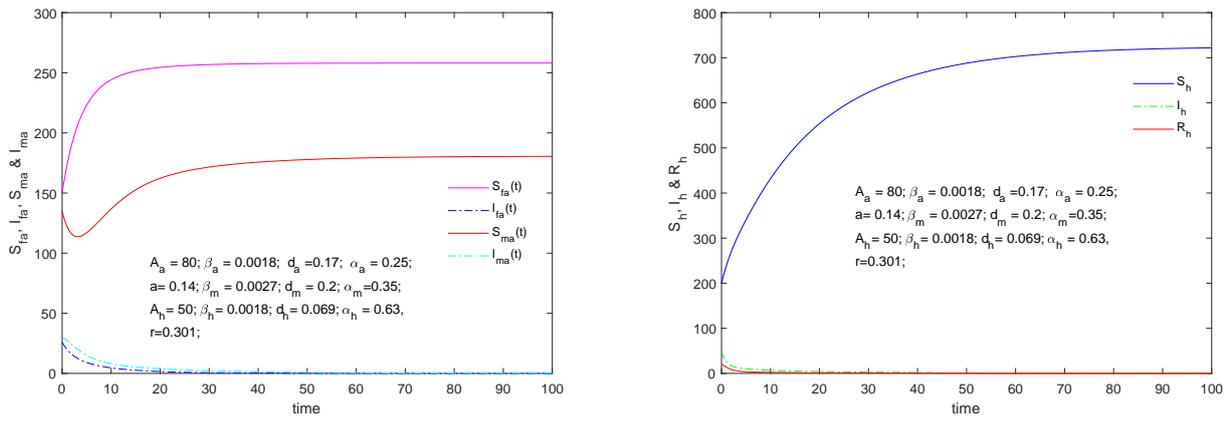


Figure 1. Time-variation diagram of system (2-8) when  $R_0 = 0.88680353 < 1$ .

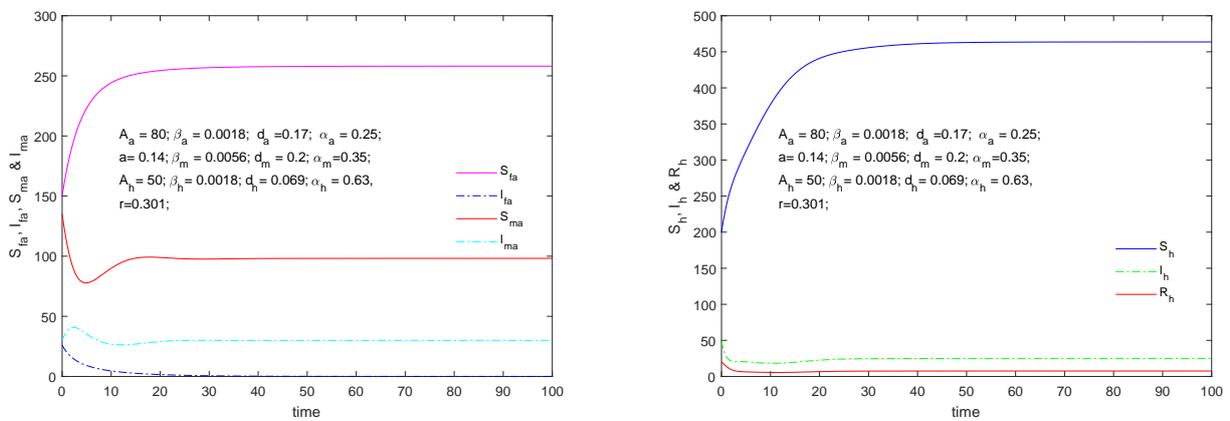


Figure 2. Time-variation diagram of system (2-8) when  $R_{01} = 0.82949309 < 1$  and  $R_{02} = 1.83929618 > 1$ .

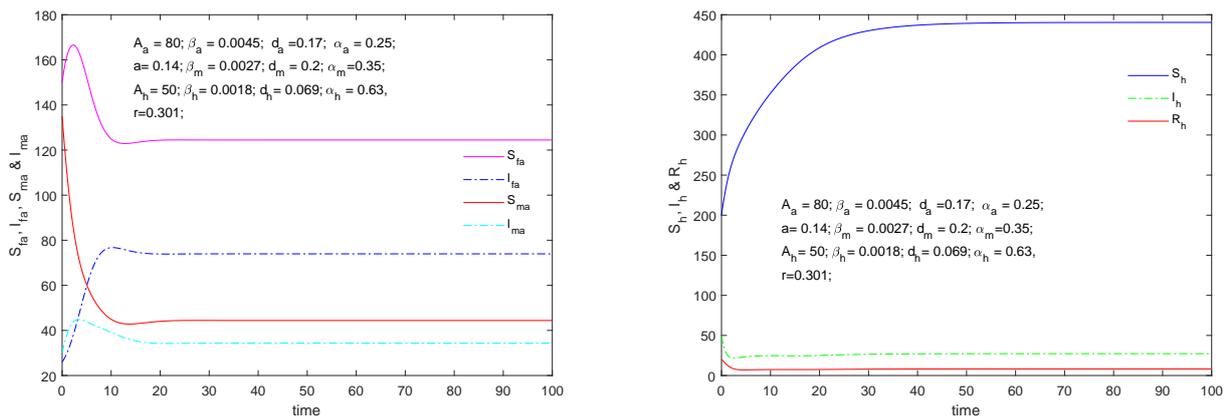


Figure 3. Time-variation diagram of system (2-8) when  $R_{01} = 2.07373271 > 1$  and  $R_{02} = 1.83929618 > 1$ .



**Table 4.** Table showing the effect of  $\beta_m$  on the reproduction number

$\beta_m$	0.0056	0.1556	0.3056	0.4556	0.6056	0.7556	0.9056
$R_{01}$	2.0737	2.0737	2.0737	2.0737	2.0737	2.0737	2.0737
$R_{02}$	1.8393	51.1062	100.3730	149.6399	198.9067	248.1736	297.4405

4. If  $R_{01} > 1$  and  $R_{02} < 1$ , (i.e.  $R_0 > 1$ ), there exists no positive equilibrium.

In the case of reinfection, as included in this paper, it is observed that the number of susceptible human increase more than the case of no re-infection (as the case in [1]). This can be traced to the fact that a fraction of the recovered human (i.e. those who do not die naturally) move back into the susceptible class. In order to reduce the spread of the avian influenza, it is observed from Table 1, Table 2, Table 3, Table 4 and Figure 4 that the following measures can be taken;

1. increase  $\alpha_a$  and  $\alpha_m$ . This can be achieved by killing of infected poultry.
2. reduce  $\beta_a$ ,  $\beta_m$  and  $\beta_h$ . This can be achieved by closing down some farms and markets where infection is detected.

It is important to note that our results agree with the results obtained in [1].

The two authors jointly developed the model. The first author did the dynamical analysis and the second author did the simulation and arranged the document.

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